International Journal of Theoretical Physics, Vol. 32, No. 7, 1993

Partial-State Formalism and the Object–Subject Split in Quantum Mechanics

Fedor Herbut[1]

Received December 21, 1992

A suitable general statistical framework is established by taking quantum mechanics as full, and other (state-distinguishing) statistical theories as partial theories with respect to a given relevant subset of observables. The partial theory exists and is unique up to equivalence. The choice of the simplest or canonical one is determined. The recently introduced hybrid, i.e., half quantum mechanical and half classical discrete, statistical states obtain thus their rightful place in a hierarchy of relevant quantum statistical theories. On the other hand, these states are shown to represent a derivation of the quantum object–subject split with a well-defined subject that encompasses preparator or measuring instrument in a natural way.

1. INTRODUCTION

The state of an object is usually described by a state vector or more generally by a statistical operator. By this it is understood that the world is divided into two (nonoverlapping) parts: the object and the rest of the world, and that the quantum mechanical formalism says nothing about the latter. This is the usual *split* (frequent synonyms: cut, division), and one can say that it has an empty or ill-defined subject.

It seems to me that this state of affairs is fully satisfactory only in the frame of ideas of the original Copenhagen interpretation (Stapp, 1972), which might be called, as suggested by Shimony's (1963) discussion, "early Bohr." It can also be called instrumentalism, because the instruments, first of all the preparator (of the mentioned state), are, by postulate, outside the formalism and are only empirically recognizable macroscopic objects in the laboratory.

Bell (1990) (cf. also Gottfried, 1991) rightly criticizes the idea of the split on the basis of the following (freely formulated) requirements:

[1]Faculty of Physics, University of Belgrade, P.O. Box 550, 11001 Belgrade, Yugoslavia.

0020-7748/93/0700-1153$07.00/0

(i) A satisfactory theory should have as few interpretative elements outside the formalism as possible.

(ii) The interpretative elements inside the theory should follow from the latter in a logical way and in a manner as natural as possible.

The *first aim* of this article is to show that Bell's two requirements can be satisfactorily met as far as the preparator and the measuring instrument as "subject" are concerned. The *second aim* of this work is to introduce and elaborate the *partial-state* concept (as a natural quantum mechanical notion), and to apply it to the realization of the first aim.

To see how to replace the criticized empty-subject idea by relevant information, we start with a quotation from what Shimony's (1963) discussion suggests to call "the late Bohr" (1961, p. 50): "The main point here is the distinction between the *object* under investigation and the *measuring instruments* which serve to define, in classical terms, the conditions under which the phenomena appear." And a few lines below: "heavy bodies like diaphragms and shutters ..., in contrast to the proper measuring instruments, these bodies together with the particles would constitute the system *to which the quantum mechanical formalism has to be applied*" (italics added).

The first sentence can be ascribed also to "early Bohr," or the original Copenhagen interpretation. What is new in the "late Bohr" is expressed in the second sentence: in some cases macroscopic instruments can be viewed as quantum objects. This suggests we start by "applying the quantum mechanical formalism," as Bohr put it, to object + subject ($O+S$), and then to shift the split so that S becomes the subject. We denote the *split* (before or after the shift) as O/S, where "/" stands for the *cut* between object and subject. (Note that we use the term "cut" not as a synonym for split, but as a part of the latter.)

Applicability of the quantum mechanical formalism to $O+S$ in the initial ($O+S$) situation implies that all quantum mechanical measurements are, in principle, performable on $O+S$. Contrariwise, after the shift (toward the object), the new subject S is described classically, and this corresponds to the fact that only macroscopic measurements should be performable on it.

"Macroscopic measurements" were defined by von Neumann (1955, Chapter V, Section 4) by a suitable compatible set of quantum observables. (Among them were observables approximating the coordinate and the conjugate linear momentum.) It is a mathematical fact that such a set can be rewritten as consisting of functions of one observable. We refer to this observable as *basic*, and, for simplicity, we assume (like von Neumann did) that it has a purely discrete spectrum.

This brings us essentially to the Jauch (1964, 1968) approach to quantum measurement theory [cf. also Herbut, 1986a). It will be utilized in Section 3 in the hybrid states.

In Section 2 we introduce a sufficiently wide framework of general statistical ideas to enable us to derive the general partial-state concept. In Section 4 we return to discussing the split, in particular to shed light on the roles of preparator and measuring instrument.

Finally, Section 5 contains some speculations on the possible applicability of the partial-state notion to enlargement of quantum mechanics when considered as incomplete.

2. PARTIAL STATES

One gets the first intuitive glimpse of the concept of a *partial state* by taking the state of a subsystem: Let ρ_{12} be the state of a composite quantum system, mathematically, a statistical operator in a tensor-product Hilbert space $\mathscr{H}_1 \otimes \mathscr{H}_2$. Then, restricting onself to the first-subsystem observables, i.e., to Hermitian operators of the form $A_1 \otimes 1$, one has the well-known relations

$$\mathrm{Tr}_{12}(\rho_{12}(A_1 \otimes 1)) = \mathrm{Tr}_1(\mathrm{Tr}_2 \rho_{12}) A_1 \equiv \mathrm{Tr}_1 \rho_1 A_1 \tag{1}$$

where Tr_i is the partial trace over subsystem $i = 1, 2$, and the statistical operator

$$\rho_1 \equiv \mathrm{Tr}_2 \rho_{12} \tag{2}$$

describes the state of the first subsystem.

Since the set of all first-subsystem observables is a subset of the set of all observables for the composite system, the state ρ_1 of the first subsystem performs part of the task of the full state ρ_{12}: it predicts the averages for first-subsystem observables. In this sense we may call ρ_1 a partial state.

A specific feature of this case is the fact that also the partial state is, in its turn, a quantum state (a statistical operator in $\mathscr{H}_1$). In Section 3 we show that there exist also partial states of a different kind.

The first purpose of this section is to define the general concept of a partial state and that of a preorder between partial states (giving rise to chains). We do this in the framework of general statistical ideas. We start by introducing some elementary and standard notions and terms.

Definition 1. Let I be a natural number, and let $\{w_i\colon i = 1, 2, \ldots, I\}$ be an ordered set of positive numbers adding up to one. Then we have a set of *statistical weights*. If $\{w_i\colon w_i > 0,\ i = 1, 2, \ldots,\ I, \sum_i w_i = 1\}$ is a set of statistical weights, and $\{s_i: i=1, 2, \ldots, I\}$ $(\subseteq S)$ is a corresponding set of elements, then

$$s \equiv \sum_i w_i s_i \in S$$

is called a *convex-linear combination* of elements if the operation satisfies the following requirement: If each state s_i is given as a convex-linear combination $s_i = \sum_{k(i)} w_{k(i)} s_{k(i)}$ in its turn, then

$$s = \sum_i \sum_{k(i)} (w_i w_{k(i)})\, s_{k(i)}$$

is also a convex-linear combination (with double-indexed statistical weights and states). It the convex-linear combination always exists, then S is said to be a *convex* set. Finally, let $\langle v, s \rangle$ be a formula associating a number with each element $(v, s) \in (V \times S)$. It is said to be *convex linear* in S if for every convex-linear combination in S, and every $v \in V$, we have

$$\left\langle v, \sum_i w_i s_i \right\rangle = \sum_i w_i \langle v, s_i \rangle$$

We define a general state-distinguishing statistical theory.

Definition 2. A *statistical theory* $\{V, S, \langle v, s \rangle\}$ consists of a set V of *variables* (each having a set of values), of a convex set S of *states*, and of a formula $\langle v, s \rangle$ giving, for each state $s \in S$, the *average* of each variable $v \in V$. (The average may turn out to be infinite for some pairs of variable and state.) This expression has to be convex linear in the states. Besides, the theory has its *empirical part*, consisting of a set of laboratory procedures allowing the measurement of each variable (in any state), and of a set of laboratory procedures making possible the preparation of laboratory ensembles for each state. The measurements are performed on the individual systems in these ensembles. The average-value formula gives the (finite or infinite) real number equalling (the limit value of) the arithmetical mean of the measured values in the laboratory ensemble (when the number of systems in it tends to infinity). Finally, we call a statistical theory *state distinguishing* if

$$(\langle v, s \rangle = \langle v, s' \rangle, \forall v \in V) \Rightarrow s = s' \tag{3}$$

i.e., if distinct states must differ in the average of at least one variable.

We restrict ourselves to state-distinguishing statistical theories throughout.

The convex structure of the set of states S corresponds to the empirical procedure of *mixing* (or taking the union) of laboratory ensembles, which, by definition, has to be done in constant proportions (of the numbers of systems in the ensembles). Upon normalization to one, these proportions become statistical weights, which go into the definition of convex combinations (making up the convex structure of S). The requirement of convex linearity on the average-value formula is an immediate consequence of the empirical facts that a convex combination means mixing and that the formula giving the average actually provides one with the arithmetical mean result in the ensemble.

Next we define *partial theories* and *partial states.*

Definition 3. Let there be given two statistical theories $\{V, S, \langle v, s\rangle\}$ and $\{W, \Sigma, \langle w, \sigma\rangle\}$ with two maps from the first to the second one: a bijection (we call it *the map of variables*) of a *subset* V' ($\subseteq V$) onto W, and a surjection (we call it *the map of states*) of S onto Σ such that the latter preserves the convex combinations. Let, further, the set of values of each variable $v \in V'$ be an invariant of the map of variables and of its inverse, and let the average value be an invariant of the two maps (restriction to V' is understood). Further, let the variables that are counterparts by the map of variables be measured by the same laboratory procedures. Finally, let the preparations of laboratory ensembles for the states of the first theory coincide with the preparations of ensembles for the states (of the second theory) that correspond (by the map of states) to the former. Then we say that the second theory is a *partial theory* with respect to the first one and with respect to V', and that the states of the second statistical theory are *partial states* (with respect to V') of those states of the first theory of which the images (by the map of states) they are.

The requirements are obviously necessary in view of the above outlined empirical meaning of the statistical entities involved. Further, the states of a subsystem mentioned in the Introduction satisfy Definition 2 (cf. Proposition 2 below).

Definition 4. If in the relation between two statistical theories that is described in Definition 3 the subset V' of variables in the first statistical theory is the entire set V of variables and the surjection (of states) is a bijection, then the maps establish a symmetrical relation. We speak in this case of *equivalent statistical theories* and of *equivalent statistical states* (and of *same* variables).

The just defined concept of equivalence, of course, generalizes that of identity.

Let us imagine along with quantum mechanics of the given quantum system other statistical theories forming a sufficiently wide set with the binary relation defined in Definition 3 in it. This relation is obviously reflexive and transitive, i.e., a so-called *preorder* (Birkhoff, 1940). As known (and as it is straightforward to see), every preorder *induces an equivalence relation:* two elements being, by definition, equivalent if they stand in the preorder relation whichever of the two elements one takes first. In the quotient set (the elements of which are the equivalence classes) the preorder induces (via arbitrary element) and order relation, i.e., a preorder with the additional property of antisymmetry: the equivalence defined by this preorder is the identity in the quotient set.

Envisaging the set of all states in all the statistical theories, the above partial-state relation is also a preorder (transitivity is easily established). This implies that we can have *chains* of substates, and *equivalent* substates, defining equivalence via the preorder as explained above. We write the preorder (of theories and states) as "<" in case of inequivalence.

It is useful to find among equivalent partial states simplest or canonical ones. A statistical theory is *canonical* if its entities (the variables, the states, and the average value formula) have minimal redundancy in their form, i.e., if they contain all the indispensable information and nothing else. This gives the basic idea, but by itself it is not sufficiently specific. We give a more specific definition of canonicity below.

Following Jauch (1964, 1968), we now incorporate a known general procedure into our concept of partial states.

Definition 5. Let V' be an arbitrary given subset of variables in a statistical theory $\{V, S, \langle v, s\rangle\}$ (we refer to to the latter as the initial theory). We define the following *equivalence relation* $\sim_{V'}$ in the set S of all states of the theory:

$$s, s' \in S, \qquad s \sim_{V'} s' \qquad \text{if} \quad \langle v, s\rangle = \langle v, s'\rangle \qquad \text{for every} \quad v \in V'$$

Let us denote by $S/\sim_{V'}$ the corresponding quotient set, i.e., the set of all equivalence classes in S.

Definition 6. Utilizing the concepts of Definition 5, we now define the requisite entities for what we shall call *the statistical theory of equivalence classes* induced by the subset V' of variables from the initial theory:

The set of variables is V' with the same value sets and measurement procedures as in the initial statistical theory. The set of states is $S/\sim_{V'}$. For every $C \in S/\sim_{V'}$ all the preparation procedures of all $s \in C$ make up, by definition, the preparation procedures of the equivalence class C. A convex structure is induced in $S/\sim_{V'}$ by that in S via arbitrary representatives: Let $\{w_i: i=1, 2, \ldots, I\}$ with $I=2$, or 3, or ...., be a set of statistical weights, and $\{C_i: i=1, 2, \ldots, I\}$ a set of states from $S/\sim_{V'}$. Then, by definition

$$S/\sim_{V'} \ni C \equiv \sum_{i=1}^{I} w_i C_i \equiv C\left(\sum_{i=1}^{I} w_i s_i\right)$$

where $C(s)$ denotes the equivalence class to which the statistical states s belong, and $\{s_i: s_i \in C_i, i=1, 2, \ldots, I\}$ are arbitrary elements from the corresponding classes. The average value expression is given by

$$\forall v \in V', \quad \forall C \in S/\sim_{V'}: \qquad \langle v, C\rangle \equiv \langle v, s\rangle, \quad s \in C$$

where we have the average defined in the initial theory on the rhs, and s is an arbitrary element of C.

It is easy to show that what we have called the statistical theory of equivalence classes, i.e., $\{V', S/\sim_{V'}, \langle v, C\rangle\}$, satisfies the requirements for a state-distinguishing statistical theory (cf. Definition 2). In Jauch's (1964, 1968) approach the equivalence classes with a suitable V' played an important role in his attempt at a resolution of the quantum measurement problem. He called these classes "macrostates," thinking of them as of states of macrosystems or of classical systems.

It should be pointed out that it may happen that we have a set of variables V' that is a *proper* subset of the set V of all variables in the initial theory, and still we can have all equivalence classes containing a single element, i.e., we can actually reobtain the initial theory unchanged (up to equivalence). Historically, a very important example for this possibility appeared in the context of distant correlations in quantum mechanics. A well known reaction to the famous Einstein, Podolsky, and Rosen (1935) article was Furry's (1936) paper, in which it was proved that if one restricts oneself to coincidence observables (as one must, due to the distantness of the particles), in terms of this restricted set V' one can still distinguish any two two-particle statistical operators, i.e., all equivalence classes have a single element.

The following result is easily proved.

Proposition 1. Let $\{V, S, \langle v, s\rangle\}$ be a statistical theory and V' a subset of variables in it. Another state-distinguishing statistical theory gives partial states with respect to the former theory and with respect to V' if and only if it is *equivalent to the statistical theory of equivalence classes* $\{V', S/\sim_{V'}, \langle v, C\rangle\}$ induced from the former theory by V'.

We have an immediate consequence of Proposition 1:

Theorem 1. When a statistical theory $\{V, S, \langle v, s\rangle\}$ and a subset V' $(\subseteq V)$ are given, the corresponding state-distinguishing partial statistical theory always *exists* and it is *unique* up to equivalence.

Making use of the statistical theory of equivalence classes, we now give a more specific (and more practical) criterion for canonicity.

Definition 7. Let $\{W, \Sigma, \langle w, \sigma\rangle\}$ be a state-distinguishing partial statistical theory with respect to $\{V, S, \langle v, s\rangle\}$ (the "initial theory") and with respect to V' $(\subseteq V)$. The former theory can be considered *canonical* if each of its states is made up of entities that are determined by the states of the initial theory so that these entities are necessary and sufficient for the corresponding equivalence class ("corresponding" in the sense of the equivalence in Proposition 1).

Let us turn now to quantum mechanics (QM).

Lemma 1. Quantum mechanics is a state-distinguishing statistical theory. Writing the observables (Hermitian operators) as A and the states (statistical operators) as ρ, we have

$$\{V, S, \langle v, s\rangle\} \equiv \{\{\text{all } A\}, \{\text{all } \rho\}, \operatorname{Tr} A\rho\}$$

Proof. The only nontrivial part of the proof is perhaps the claim that the theory is state distinguishing. Let

$$\forall A: \quad \operatorname{Tr} A\rho = \operatorname{Tr} A\rho'$$

Confining ourselves to elementary events (ray projectors) $A \equiv |\psi\rangle\langle\psi|$, we obtain

$$\forall |\psi\rangle \in \mathscr{H}, \quad \langle\psi|\psi\rangle = 1: \qquad \langle\psi|\rho|\psi\rangle = \langle\psi|\rho'|\psi\rangle$$

which, as is well known. entails $\rho = \rho'$. ■

Finally, we return to the subsystem state ρ_1 given in (2).

Proposition 2. Let us consider quantum mechanics in two versions: as the statistical theory

$$\{V, S, \langle v, s\rangle\} \equiv \{\{\text{all } A_{12}\}, \{\text{all } \rho_{12}\}, \operatorname{Tr}_{12} A_{12}\rho_{12}\}$$

of a composite system with the state space $\mathscr{H}_1 \otimes \mathscr{H}_2$ (we call it the composite QM), and as the statistical theory

$$\{W, \Sigma, \langle w, \sigma\rangle\} \equiv \{\{\text{all } A_1\}, \{\text{all } \rho_1\}, \operatorname{Tr}_1 A_1\rho_1\}$$

of the first subsystem (we call it the subsystem QM). The underlying inclusion relation between sets of variables now has the form

$$\{\text{all } (A_1 \otimes 1)\} \equiv V' \subset V \equiv \{\text{all } A_{12}\} \tag{4}$$

Starting with the subset V' of variables, we define the map of variables (cf. Definition 3) from the composite QM to the subsystem QM as $(A_1 \otimes 1) \Rightarrow A_1$, and the map of states as $\rho_{12} \Rightarrow \rho_1 \equiv \mathrm{Tr}_2\, \rho_{12}$ [cf. (2)]. Thus, the subsystem states ρ_1 become *partial states* of the states ρ_{12} of the composite system:

$$\rho_1 < \rho_{12}$$

Proof. Straightforward [in view of (2)]. ∎

If the first subsystem is composite in its turn, i.e., if in $\mathscr{H}_1 \otimes \mathscr{H}_2$ we replace $\mathscr{H}_1$ by $\mathscr{H}_0 \otimes \mathscr{H}_1$ to obtain $\mathscr{H}_0 \otimes \mathscr{H}_1 \otimes \mathscr{H}_2$, then we have the *chain* of partial states (in this case subsystem states)

$$\rho_0 < \rho_{01} < \rho_{012}$$

where $\rho_0 \equiv \mathrm{Tr}_1\, \rho_{01} = \mathrm{Tr}_{12}\, \rho_{012}$ and $\rho_{01} \equiv \mathrm{Tr}_2\, \rho_{012}$. This is an obvious corollary of Proposition 2.

3. THE HYBRID STATES AS PARTIAL STATES

In the wake of Jauch's (1964, 1968) search for a solution of the measurement (or objectification) problem (Busch *et al.*, 1991), the statistical theory of hybrid stages was derived in a systematic and natural way in previous work (Herbut, 1986b). We give anew a definition of it in this section and we show that this theory provides us with partial states.

We have a composite quantum system with the Hilbert space $\mathscr{H}_1 \otimes \mathscr{H}_2$. Let there be given in $\mathscr{H}_2$ a discrete observable in spectral form as

$$B_2^0 \equiv \sum_k b_k^0 Q_2^{(k)} \tag{5}$$

($k \neq k' \Rightarrow b_k^0 \neq b_{k'}^0$, and k takes up the values from a given finite or countably infinite index set). It is called the *basic observable.*

We have in mind a restricted set V' of variables in $\mathscr{H}_1 \otimes \mathscr{H}_2$:

$$V' \equiv \left\{\text{all}\left(A_1 \otimes \left(\sum_k b_k Q_2^{(k)}\right)\right) : \text{any } A_1, \text{any } b_k\right\} \tag{6}$$

where A_1 is an arbitrary first-subsystem observable, and b_k are arbitrary real numbers.

Restriction to the set V' of observables does not amount to the use of statistical operators (in some Hilbert space other than $\mathscr{H}_1 \otimes \mathscr{H}_2$) as in the case of a subsystem (cf. the Introduction and Proposition 2). But the wider scheme of partial states introduced in Section 2 enables one to achieve an analogous aim.

For the initial statistical theory we take quantum mechanics of the composite system:

$$\{V, S, \langle v, s\rangle\} \equiv \{\{\text{all } A_{12}\}, \{\text{all } \rho_{12}\}, \mathrm{Tr}_{12} A_{12}\rho_{12}\}$$

The subset of variables V' is defined by (6).

We define what will turn out to be a state-distinguishing partial statistical theory $\{W, \Sigma_H, \langle w, \sigma\rangle\}$ on this basis, and we call it the statistical theory of *hybrid states* (Herbut, 1986b):

(i) The set of *variables* W consists of all entities of the form

$$w \equiv \{(A_1, b_k): \forall k\} \tag{7}$$

where A_1 is an arbitrary Hermitian operator in $\mathscr{H}_1$, and b_k are arbitrary real numbers.

(ii) The set of states Σ_H is the set of all entities

$$\sigma \equiv \{(\delta(p_k > 0)\rho_1^{(k)}, p_k): \forall k\} \tag{8}$$

where $\delta(p_k > 0)$ equals 1 if $p_k > 0$, and it is zero otherwise, $\rho_1^{(k)}$ are arbitrary statistical operators in $\mathscr{H}_1$ (unless $p_k = 0$, when $\rho_1^{(k)}$ is undefined), and $\{p_k; \forall k\}$ are arbitrary classical discrete probability distributions, i.e., $\forall k$: $p_k \geqslant 0$, $\sum_k p_k = 1$.

(iii) The *average-value* formula is given by

$$\langle w, \sigma\rangle \equiv \sum_k p_k b_k \,\mathrm{Tr}_1(A_1\rho_1^{(k)}) \tag{9}$$

The additional details that are required go as follows:

(i′) The *value set* of a variable $w = \{(A_1, b_k): \forall k\}$ is

$$\{ab_k: a \in (\text{spectrum if } A_1), \forall k\} \tag{10}$$

The measurement procedure consists in coincidence measurement of A_1 on the first subsystem and of $\sum_k b_k Q_2^{(k)}$ on the second. The value ab_k is the product of the results of these two measurements.

(ii′) Let $\{w_i > 0\colon\ i = 1, 2, \ldots, I; \sum_i w_i = 1\}$ and $\{\sigma_i\colon i = 1, 2, \ldots, I\}$ be arbitrary finite sets of statistical weights and hybrid states, respectively. More explicitly,

$$\sigma_i \equiv \{((\delta(p_k^{(i)} > 0)\rho_1^{(k,i)},\ p_k^{(i)})\colon \forall k\}, \qquad i = 1, 2, \ldots, I$$

Then the corresponding *convex combination* is, by definition, given by

$$\sigma \equiv \sum_i w_i \sigma_i \equiv \left\{\left(\delta(p_k > 0)\rho_1^{(k)},\ p_k \equiv \sum_i w_i p_k^{(i)}\right)\colon \forall k\right\} \tag{11a}$$

where

$$\rho_1^{(k)} \equiv \sum_i w_i^{(k)} \rho_1^{(k,i)}, \qquad \forall k \tag{11b}$$

with the auxiliary entities

$$w_i^{(k)} \equiv w_i p_k^{(i)}/p_k, \qquad \forall k, \quad i = 1, 2, \ldots, I \tag{11c}$$

It is straightforward to ascertain that the operation defined by (11a)–(11c) satisfies the requirement in Definition 1.

As seen from the expression for p_k in (11a) and from (11c), if $p_k > 0$, then $\{w_i^{(k)}\colon i = 1, 2, \ldots, I\}$ is a sequence of statistical weights, and if $p_k = 0$, then $p_k^{(i)} = 0$, $i = 1, 2, \ldots, I$, and $\{w_i^{(k)}\colon i = 1, 2, \ldots, I\}$ is a sequence of zeros [making $\rho_1^{(k)}$ in (11b) zero consistently with (8)].

As to the preparation procedures to obtain the hybrid states $\sigma \in \Sigma_H$, they are the same as those of the corresponding quantum states ρ_{12} (statistical operator in $\mathcal{H}_1 \otimes \mathcal{H}_2$), where "corresponding to σ" means "taken into σ by the map of states" (see Theorem 2 below).

(iii′) The average-value expression (9) is *convex linear* in the hybrid states. This is a straightforward consequence of (11a)–(11c):

$$\begin{aligned}
\left\langle w, \sum_i w_i \sigma_i \right\rangle &= \sum_k p_k b_k \operatorname{Tr}_1\left(\left(\sum_i w_i^{(k)} \rho_1^{(k,i)}\right) A_i\right)\\
&= \sum_k p_k b_k \operatorname{Tr}_1\left(A_1 \sum_i (w_i p_k^{(i)}/p_k)\rho_1^{(k,i)}\right)\\
&= \sum_i w_i \left(\sum_k p_k^{(i)} b_k \operatorname{Tr}_1(A_1 \rho_1^{(k,i)})\right)\\
&= \sum_i w_i \langle w, \sigma_i \rangle
\end{aligned}$$

The notion of a subsystem state $\rho_1 \equiv \operatorname{Tr}_2 \rho_{12}$ [cf. (2)] has, besides the partial-state concept, one more generalization (in a different "direction"):

that of a conditional state. We outline this concept here because we need it to fulfil the purpose of this section.

If ρ_{12} is an arbitrary state of a composite system, and Q_2 an arbitrary event for the second subsystem (projector in $\mathscr{H}_2$) such that $p \equiv (\mathrm{Tr}_{12}\, \rho_{12}(1 \otimes Q_2)) > 0$, then

$$\rho_1^c \equiv p^{-1}\, \mathrm{Tr}_2\, \rho_{12}(1 \otimes Q_2) \tag{12}$$

is the *conditional state* of the first subsystem under the *condition* that the event $(1 \otimes Q_2)$ has occurred (has been measured) on the composite system in the state ρ_{12}.

This interpretation of ρ_1^c follows from three facts:

(i) That ρ_1^c is a statistical operator in $\mathscr{H}_1$ (a first-subsystem state).

(ii) That the probability of the (immediate) succession of two subsystem events P_1 after Q_2 equals the probability of their coincidence in ρ_{12}.

(iii) That one has

$$\mathrm{Tr}_{12}\, \rho_{12}(P_1 \otimes Q_2) \equiv p\, \mathrm{Tr}_1\, \rho_1^c P_1 \tag{13}$$

which is the quantum counterpart of the well-know definition of conditional probability in classical probability theory and which is straightforward to derive (Herbut, 1986b).

In the special case of $Q_2 = 1$, one has $\rho_1^c = \rho_1 \equiv \mathrm{Tr}_2\, \rho_{12}$. Hence, (12) is a generalization of this formula (but it is not a partial state).

Now we can carry out the main task of this section.

Theorem 2. The *hybrid states* $\sigma \in \Sigma_H$ defined in (8) are *partial* states of some quantum state ρ_{12} (of the composite system) with respect to the *subset of variables* given by (6). By this the map of variables is determined by

$$A_1 \otimes \left(\sum_k b_k Q_2^{(k)} \right) \to w \equiv \{(A_1, b_k) : \forall k\} \tag{14}$$

and the map of states is given by

$$\rho_{12} \to \sigma \equiv \{(\delta(p_k > 0)\rho_1^{(k)}, p_k) : \forall k\} \tag{15a}$$

where

$$\forall k: \quad p_k \equiv \mathrm{Tr}_{12}\, \rho_{12}(1 \otimes Q_2^{(k)}) \tag{15b}$$

and

$$\forall k,\ p_k > 0: \quad \rho_1^{(k)} \equiv \rho_1^c \equiv p_k^{-1}\, \mathrm{Tr}_2\, \rho_{12}(1 \otimes Q_2^{(k)}) \tag{15c}$$

Proof. According to Proposition 1, Theorem 2 is true if and only if there exists a corresponding equivalence of the statistical theory of hybrid states and the statistical theory of equivalence classes. Precisely this is proved in the proof of Theorem 4 of Herbut (1986b). ■

The proof of equivalence just mentioned also implies that the statistical theory of hybrid states is *state distinguishing* (like that of the corresponding equivalence classes).

Theorem 3. The statistical theory $\{W, \Sigma_H, \langle w, \sigma \rangle\}$ of hybrid states is *canonical.*

Proof. In view of Definition 7, the canonicity of the theory at issue was proved in Herbut (1986b) (see Theorem 3 and Corollary 5 there). ■

We have the following chain of subsets of variables in $\mathscr{H}_1 \otimes \mathscr{H}_2$:

$$\{\text{all } (A_1 \otimes 1)\} \subset \left\{A_1 \otimes \left(\sum_k b_k Q_2^{(k)}\right): \text{any } A_1, \text{any } b_k\right\} \subset \{\text{all } A_{12}\} \quad (16)$$

because $1 = \sum_k Q_2^{(k)}$. We establish now the corresponding chain of partial states.

Proposition 3. The chain of partial states that corresponds to the chain of subsets of variables (16) is

$$\rho_1 < \sigma < \rho_{12} \quad (17)$$

where ρ_{12} is any composite-system state (it is a partial state in an improper sense, i.e., it is a full state), σ is its hybrid state in the sense of Theorem 1, and ρ_1 is its first-subsystem state:

$$\rho_1 \equiv \sum_k p_k \rho_1^{(k)} = \mathrm{Tr}_2\, \rho_{12} \quad (18)$$

[cf. (15a)–(15c)]. We can say that the hybrid state is *interpolated* between the state of the subsystem and the full state of the composite system [in the sense of (17)].

Proof. To prove $\rho_1 < \sigma$ in (17), we show that Definition 3 is applicable to this case. Let the statistical theory of hybrid states and that of quantum mechanics in $\mathscr{H}_1$ be the two statistical theories required. Let $V' \equiv \{\{(A_1, 1): \forall k\}: \text{any } A_1\}$ be the given subset of the set V of all variables in the initial theory [cf. (7)], and let $\{(A_1, 1): \forall k\} \Rightarrow A_1$ be the required bijection of variables. Further, let

$$\sigma \Rightarrow \rho_1 \equiv \sum_k p_k \rho_1^{(k)} \quad (19)$$

be the required map of states, where $\sigma \in \Sigma_H$, p_k, and $\rho_1^{(k)}$ are given by (15a)–(15c). To help to prove that it preserves convex combinations, we prove (19) first [utilizing (18)]:

$$\mathrm{Tr}_2\, \rho_{12} = \sum_k \mathrm{Tr}_2(\rho_{12}(1 \otimes Q_2^{(k)})) = \sum_k p_k \rho_1^{(k)}$$

[cf. (15b), (15c)]. Thus, evaluating ρ_1 directly from ρ_{12} and via the hybrid state that is defined by ρ_{12}, one obtains the same result.

Now, let $\{\sigma_i : i = 1, 2, \ldots, I\}$ and $\{w_i : i = 1, 2, \ldots, I\}$ be an arbitrary set of hybrid states and an arbitrary set of positive statistical weights, respectively. We have to prove that if

$$\sigma_i \Rightarrow \rho_1^{(i)}, \qquad i = 1, 2, \ldots, I$$

then $\sigma \equiv \sum_i w_i \sigma_i \Rightarrow \rho_1 \equiv \sum_i w_i \rho_1^{(i)}$. Let us have [in the sense of the second partial order in (17) and Definition 2]

$$\rho_{12}^{(i)} \Rightarrow \sigma_i, \qquad i = 1, 2, \ldots, I$$

and we define $\rho_{12} \equiv \sum_i w_i \rho_{12}^{(i)}$. Since $\rho_{12} \Rightarrow \sigma \equiv \sum_i w_i \sigma_i$ and $\mathrm{Tr}_2\, \rho_{12} = \sum_i w_i \mathrm{Tr}_2\, \rho_{12}^{(i)}$, we have $\rho_1 = \sum_i w_i \rho_1^{(i)}$. In view of the preceding proof, we have proved the required convex-combination preservation.

To prove that the map of states is a surjection, we take an arbitrary ρ_1. Let ρ_{12} be a composite-system state, such that $\mathrm{Tr}_2\, \rho_{12} = \rho_1$, and let ρ_{12} determine σ as its hybrid state. Then σ will, in its turn, determine the same ρ_1 [as it follows from (19)].

Further, we prove that the two maps together preserve the average value:

$$\langle w, \sigma \rangle \equiv \sum_k p_k \mathrm{Tr}_1(A_1 \rho_1^{(k)}) = \mathrm{Tr}_1(A_1 \rho_1)$$

Finally, the map of variables is obviously measurement-procedure and value preserving. Since in the chain of maps of states

$$\rho_1 \Leftarrow \sigma \Leftarrow \rho_{12}$$

implied by (17) the preparation procedures of σ are by definition those of ρ_{12}, and the preparation procedures of ρ_1 equal those of ρ_{12}, the map $\rho_1 \Leftarrow \sigma$ preserves the preparation procedures as required. ■

There is another conspicuous chain of variables:

$$\left\{1 \otimes \left(\sum_k b_k Q_2^{(k)}\right) : \text{any } b_k\right\} \subset \left\{A_1 \otimes \left(\sum_k b_k Q_2^{(k)}\right) : \text{any } A_1, \text{any } b_k\right\} \subset \left\{\text{all } A_{12}\right\} \tag{20}$$

The statistical theory $\{W, S_c, \langle w, s\rangle\}$ that corresponds to the first subset is defined as follows: $W \equiv \{\text{all } w \equiv \{b_k\text{: any } b_k, \forall k\}\}$, $S_c \equiv \{p_k\text{: } p_k \geqslant 0, \forall k, \sum_k p_k = 1\}$ (the classical discrete probability distributions), and the average-value expression is $\langle w, s\rangle \equiv \sum_k p_k b_k$.

It is easy to prove that one has the following chain of partial states corresponding to (20):

$$s < \sigma < \rho_{12} \tag{21}$$

where $s \equiv \{p_k\text{: } p_k \geqslant 0, \forall k, \sum_k p_k = 1\} \in S_c$, is the image by the corresponding map of states of the hybrid state $\sigma \equiv \{(\delta(p_k > 0)\rho_1^{(k)}, p_k)\text{: } \forall k\}$ [cf. (15a)–(15c)], etc.

4. THE SPLIT IN THE CASE OF THE PREPARATOR AND THE MEASURING INSTRUMENT

Even in foundationally oriented articles on quantum mechanics one rarely discusses the preparator. This goes hand in hand with the suppressed empty-subject split mentioned in the Introduction. Namely, it is precisely the role of the preparator that is suppressed in the trivial, empty-subject split.

We now show that a description of a *split with a well-defined subject* for the case of a a *preparator* is implied by the corresponding hybrid partial state. We confine ourselves to the special case of *Stern–Gerlach measurement and preparation.* (It is straightforward to extend the description to the general case.)

Let

$$\psi(s_z, \mathbf{r}) = (1/2)^{1/2} \chi_+(s_z)\, \phi_+(\mathbf{r}) + (1/2)^{1/2}\, \chi_-(s_z)\, \phi_-(\mathbf{r}) \tag{22}$$

be the spin-spatial wave function of an electron [sitting on an atom of silver, e.g., that is suppressed in (22)] in the magnetic field of the Stern–Gerlach apparatus (Cohen-Tannoudji *et al.*, 1977, p. 395). Here $\chi_+(s_z)$ and $t_-(s_z)$ are the spin-up and spin-down states $|\pm\rangle$ in s_z representation, and $\phi_+(\mathbf{r})$ and $\phi_-(\mathbf{r})$ describe upward-deflected and downward-deflected electrons, respectively.

The simplest way to treat the s_z *measurement* and *preparation* is (Peierls, 1985) to take the empty-subject *split* as follows (see the Introduction):

$$O + S \equiv \text{spin} + (\text{spatial degree of freedom})$$

To fill in the required information for an explicit definition of the subject, we have to specify *the basic observable* (5). Instead, one can take a certain function of the basic observable, the so-called pointer observable (Herbut, 1991). For simplicity, we take these two concepts as equal.

In our case we can take $\{+, -\}$ for the set of values of the index k, and define

$$Q_2^{(+)} \equiv \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} |\mathbf{r}\rangle\langle\mathbf{r}| \, dx \, dy \, dz$$

$$Q_2^{(-)} \equiv \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{0} |\mathbf{r}\rangle\langle\mathbf{r}| \, dx \, dy \, dz$$

i.e., the projectors corresponding to the upper and the lower half-spaces (and for b_+ and b_- any two distinct real numbers). Putting $\rho_{12} \equiv |\psi\rangle_{12}\langle\psi|_{12}$, where $|\psi\rangle_{12}$ is given in s_2, $\mathbf{r}$ representation by (22), we can apply (15a)–(15c) to evaluate *the corresponding hybrid partial state.* We obtain

$$\sigma = \{(|+\rangle\langle+|, 1/2), (|-\rangle\langle-|, 1/2)\}$$

Evidently, this hybrid state implies (or is adapted to) the *split*:

$$O/S \equiv \text{spin}/(\text{spatial degree of freedom})$$

Utilizing the convex structure in Σ_H given by (11a)–(11c), it is straightforward to prove the following general result.

Lemma 2. *Any* hybrid state $\sigma \in \Sigma_H$ [cf. (8)] can be *uniquely decomposed* into states *each with a sharp value of the basic observable*:

$$\sigma = \sum_k p_k \{(\delta_{k',k}\rho_1^{(k)}, \delta_{k',k}) : \forall k'\} \tag{23}$$

Thus, we can write our hybrid state as the following *mixture*:

$$\sigma = (1/2)\{(|+\rangle\langle+|, 1), (0, 0)\} + (1/2)\{(0, 0), (|-\rangle\langle-|, 1)\} \tag{24}$$

It is noteworthy that restriction to observables of V' [cf. (6)] converts a pure full state like the one given by (22) into a partial state that is a mixture.

It is easy to show that we have two *pure* hybrid states on the rhs of (24). We discuss now the first one.

Let us imagine that we have pierced the upper plate of the Stern–Gerlach screen (Cohen-Tannoudji *et al.*, 1977) at the appropriate spot. Let the device be made so that it informs the experimenter if an electron has passed the hole, e.g., by measuring an anticoincidence consisting of the occurrence of the event that the electron has entered the apparatus and the nonoccurrence of the event that the electron has arrived at the lower plate in the expected time interval. Then we have a first-kind or repeatable *measurement* (Busch *et al.*, 1991), which is at the same time a *preparation* by measurement, both in the *selective* (i.e., definite-result-subensemble) version. This can be viewed as applying to individual systems.

We concentrate on the hybrid partial state

$$\{(|+\rangle\langle+|, 1), (0, 0)\} \tag{25}$$

that is the first constituent in the decomposition (24). Now we complete the definition of our *subject* by assuming that the quantum event $Q_2^{(+)}$ (see above) *has occurred.* Then the electron is prepared in the corresponding *conditional state* $\rho_1^{(+)} \equiv |+\rangle_1\langle+|_1$.

The symmetrical discussion applies to the second constituent in (24).

To sum up, the hybrid partial state (25), e.g., by itself fully defines the *split with a well-defined subject* as far as application to an individual system is concerned. The quantum mechanical part applies to the object, and the classical discrete part to the subject. Besides, since it is a sharp value (concerning the basic observable) state, it contains the information on which characteristic event of the basic observable, or *subject event,* has occurred.

Besides preparation by measurement, there is also *conditional preparation.* In our case of the Stern–Gerlach experiment, we obtain this if the instrument is not provided with the ability to measure the mentioned anticoincidence. We can geometrically, i.e., by confining ourselves to the upper half-space, arrange the subsequent measurements so that a result is obtained *only if* the electron had passed the hole [retroactive apparent occurrence; see Wheeler (1983); cf. also Herbut (1994)]. It seems to me that this case is in practice met more often than preparation by measurement because it is simpler.

The interesting point about conditional preparation is that the object is described by the same state, in our case by $|+\rangle$, because, as it was explained above, it is a conditional state anyway. Whether the condition [the event $Q_2^{(+)}$ in the case of the first constituent in (24)] has occurred in an ordinary way or by apparent retroaction is immaterial for the hybrid-state formalism (and also if translated into a more standard form).

Finally, one may wonder if the hybrid form of the *split with a well-defined subject* has any advantage over its *standard form.* The latter consists in three requirements:

(i) That the *cut* is between the spin and spatial degrees of freedom, making the former *object* and the latter *subject.*

(ii) That the state of the *individual* object is a *conditional state.*

(iii) That the above half-space projectors define *the basic observable*, and that *occurrence* of $Q_2^{(+)}$ is the *condition.*

The hybrid partial state form of the split (25) contains precisely the same information. The ability to use it requires some learning (perhaps a disadvantage), but it contains in a systematically organized way the essential idea of a split (with a nonempty subject).

In the language of partial states, a split with a well-defined subject is obtained from a split with an ill-defined (or empty) subject if the corresponding hybrid state σ is interpolated between the subsystem state ρ_1 and the full state ρ_{12} [cf. (17)].

We close this section by remarking that the concept of *occurrence* of the subject event $Q_2^{(k)}$, though crucial for the notion of the split with a well-defined subject when applied to individual systems, remains outside quantum mechanics. This is tantamount to the well-known problem of measurement theory or of objectification in measurement (Busch *et al.*, 1991). Namely, "occurrence" does not seem to be an idea inherent in the quantum mechanical formalism, though it appears indispensable.

5. QUANTUM STATES AS PARTIAL STATES IN INCOMPLETE QUANTUM MECHANICS

In this last section we speculate on the possible usefulness of the partial-state concept for a possible future completion of quantum mechanics either by only some *hidden variables*, or rather *beables* as Bell (1989) has called them (cf. also Herbut, 1991) (a partially causal theory), or by making all quantum observables having sharp values on the (hidden) subquantum level (a fully causal theory).

In such a completion the quantum mechanical states would be partial states. A completed subquantum (partially or fully causal) statistical theory would be the initial theory $\{V, S, \langle v, s\rangle\}$ in the notation of Section 2, whereas quantum mechanics itself would take the role of $\{W \equiv V' \equiv \{\text{all } A\}, \Sigma \equiv \{\text{all } \rho\}, \langle A, \rho\rangle \equiv \text{Tr } A\rho\}$.

It is then clear from Definition 3 that $(V \backslash V')$ would be the set of all new variables, and that the subquantum states $s \in S$ belonging to the same inverse class of the map of states could be possibly distinguished only by the averages of the variables of this set.

Summing up the procedure of completion according to Definition 3, we keep the quantum observables (essentially) unchanged, add a set of new ones, and replace each quantum state by a set of subquantum ones. This procedure can be made more intricate by replacing also the quantum observables by sets of subquantum variables. Such a program was proved possible by Durdevic (1991).

REFERENCES

Bell, J. S. (1989). Beables for quantum field theory, in *Speakable and Unspeakable in Quantum Mechanics*, Cambridge University Press, Cambridge, pp. 173–180.

Bell, J. S. (1990). *Physics World*, **1990** (August), 33.

Birkhoff, G. (1940). *Lattice Theory*, American Mathematical Society, Providence, Rhode Island.

Bohr, N. (1961). *Atomic Physics and Human Knowledge*, Science Editions, New York.

Busch, P., Lahti, P. J., and Mittelstaedt, P. (1991). *The Quantum Theory of Measurement*, Springer-Verlag, Berlin.

Cohen-Tannoudji, C., Diu, B., and Laloë, F. (1977). *Quantum Mechanics*, Vol. 1, Wiley, New York.

Durdevic, M. (1991). *Journal of Physics A: Mathematical and General*, **24**, 549.

Einstein, A., Podolsky, B., and Rosen, N. (1935). *Physical Review*, **47**, 777.

Furry, W. H. (1936). *Physical Review*, **49**, 393.

Gottfried, K. (1991). *Physics World*, **1991** (October), 34.

Herbut, F. (1986a). *International Journal of Theoretical Physics*, **25**, 863.

Herbut, F. (1986b). *International Journal of Theoretical Physics*, **25**, 1215.

Herbut, F. (1991). *Journal of Physics A: Mathematical and General*, **24**, 1785.

Herbut, F. (1994). Delayed-choice experiments and retroactive apparent occurrence in the quantum theory of measurement, *Foundations of Physics*, to appear.

Jauch, J. M. (1964). *Helvetica Physica Acta*, **37**, 293.

Jauch, J. M. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley, London.

Peierls, R. F. (1985). Observations in quantum mechanics and the "collapse of the wave function," in *Symposium on the Foundations of Modern Physics*, World Scientific, Singapore, p. 187.

Shimony, A. (1963). *American Journal of Physics*, **31**, 755.

Stapp, H. P. (1972). *American Journal of Physics*, **40**, 1098.

Von Neumann, J. (1955). *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, New Jersey.

Wheeler, J. A. (1983). Law without law, in *Quantum Theory and Measurement*, J. A. Wheeler and W. H. Zurek, eds., Princeton University Press, Princeton, New Jersey, p. 182.